Relative monads and distributors

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Motivation

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A particular motivation for the theory of distributors is its application to formal category theory. Categorical proofs based on the theory of distributors typically hold in a general 2-dimensional setting, allowing us to recover these results for various flavours of category theory at once (e.g. ordinary, enriched, internal, and so on).

Overview

1. Distributors and double categories
2. Monads and loose-monads
3. Relative monads
4. Kleisli categories
5. The pullback theorem
6. Consequences

Distributors and double categories

On the nature of category theory

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- Functors.
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What are the fundamental building blocks of category theory? In other words, what do we need to *do* category theory? Certainly we need at least the following (e.g. to define adjunctions, monads, colimits, etc.).

- Categories.
- Functors.
- Natural transformations.

However, there are many fundamental concepts in category theory that cannot be defined with just these concepts (e.g. weighted colimits, pointwise extensions, density, full faithfulness, etc.). To capture these concepts, we need one more building block.

- Distributors.

Distributors

Definition 1 (Bénabou)

Let A and B be categories. A distributor (a.k.a. profunctor or (bi)module) $A \nrightarrow B$ is a functor $B^{\mathrm{op}} \times A \to \mathbf{Set}$.

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A distributor $p \colon A \nrightarrow B$ may be thought of as a categorified notion of relation, i.e. a function $B \times A \to \{\bot, \top\}$. A helpful intuition is to think of the elements of $p(b, a)$, for each $b \in |B|$ and $a \in |A|$, as heteromorphisms from b to a. Functoriality of p then ensures that heteromorphisms in p are closed under precomposition by morphisms in B and under postcomposition by morphisms in A.

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(In more general settings, such as enriched category theory, distributors may not be defined in terms of functors, which is why we view them as a fundamentally separate concept. This is particularly crucial for formal category theory.)

Yoneda embedding as a distributor

As suggested by the definition of a distributor, to every category A there is a canonical endo-distributor on A, given by the homomorphisms of A.

Example 2

Let A be a (locally small) category. The hom-sets of A form an identity distributor $A(1,1)\colon A \nrightarrow A$, defined by

$$A(1,1)(a, a') := A(a, a')$$

(Note that the hom-set functor $A^{\mathrm{op}} \times A \to \mathbf{Set}$ is the uncurried form of the Yoneda embedding $A \to \mathbf{Set}^{A^{\mathrm{op}}}$.)

Representable and corepresentables

Every functor $f \colon A \to B$ induces two distributors.

Example 3

A representable distributor $B(1, f) \colon A \nrightarrow B$, defined by

$$B(1, f)(b, a) := B(b, fa)$$

Example 4

A corepresentable distributor $B(f, 1) \colon B \nrightarrow A$, defined by

$$B(f, 1)(a, b) := B(fa, b)$$

Restriction

The representable and corepresentable distributors associated to a functor are special cases of the following construction.

Example 5

Given every diagram of the following form,

$$
\begin{array}{ccc}
A & \xleftarrow{\ p(f,g)\ } & D \\
f \downarrow & & \downarrow g \\
C & \xleftarrow{\ \ p\ \ } & B
\end{array}
$$

there is a distributor $p(f, g) \colon A \nrightarrow D$, defined by

$$p(f, g)(d, a) := p(fd, ga)$$

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Proposition 6

Let $\ell\colon A \to B$ and $r\colon B \to A$ be functors. Then $\ell \dashv r$ if and only if there is an isomorphism of distributors:

$$B(\ell, 1) \cong A(1, r)\colon B \nrightarrow A$$

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As a consequence, we are able to define (weighted) limits and colimits, pointwise extensions, density, full faithfulness, etc. in terms of distributors.

Distributors versus presheaf categories

For ordinary categories, we can alternatively define these concepts in terms of presheaf categories. That is, for locally small categories A and B, a distributor $A \nrightarrow B$ is equivalently a functor $A \to [B^{\mathrm{op}}, \mathbf{Set}]$ by currying. However, this is not possible in general for other flavours of category theory, such as enriched category theory, where presheaf categories may not exist.

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Furthermore, using distributors allows us not to worry about size issues (e.g. taking presheaf categories of large categories).

We shall see more advantages to reasoning using distributors, in connection to the theory of monads, later in the talk.

The structure that distributors form

There are many examples of "category-like structures" that are of interest in category theory. Some examples are enriched categories, internal categories, fibred categories, indexed categories, monoidal categories, and so on.

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Axiomatising the structure of categories, functors, and natural transformations led early category theorists to the concept of 2-category [God58; Bén65; Mar65]. Enriched categories, internal categories, fibred categories, and so on, all form 2-categories.

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However, as we have mentioned, to carry out a significant amount of category theory, we also need to consider distributors. What structure, then, do categories, functors, distributors, and natural transformations form?

Virtual double categories

A virtual double category [Bur71] has a collection of objects, a collection of tight-cells $\bullet \to \bullet$ between objects, a collection of loose-cells $\bullet \nrightarrow \bullet$ between objects, and a collection of 2-cells of the following shape.

$$
\begin{array}{ccccccccc}
A_0 & \xleftarrow{\;p_1\;} & A_1 & \xleftarrow{\;p_2\;} & \cdots & \xleftarrow{\;p_{n-1}\;} & A_{n-1} & \xleftarrow{\;p_n\;} & A_n \\
f \downarrow & & & & \phi & & & & \downarrow g \\
B_0 & \xleftarrow{\hspace{4cm}q\hspace{4cm}} & & & & & & & B_n
\end{array}
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While the definition of a virtual double category may at first appear intimidating, in practice it quickly becomes intuitive to reason about them, for instance by using a string diagram calculus.

Natural transformations

A natural transformation of the form

$$A_0 \xleftarrow{p_1} A_1 \xleftarrow{p_2} \cdots \xleftarrow{p_{n-1}} A_{n-1} \xleftarrow{p_n} A_n$$

$$f \downarrow \qquad\qquad \phi \qquad\qquad \downarrow g$$

$$B_0 \xleftarrow{\qquad\qquad\qquad q \qquad\qquad\qquad} B_n$$

comprises a family of functions

$$\phi_{x_0,\ldots,x_n} \colon p_1(x_0, x_1) \times \cdots \times p_n(x_{n-1}, x_n) \to q(fx_0, gx_n)$$

for $x_0 \in |A_0|, \ldots, x_n \in |A_n|$, satisfying certain naturality laws.

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$$f \downarrow \qquad\qquad \phi \qquad\qquad \downarrow g$$
$$B_0 \xleftarrow{\hspace{6cm}}_{q} B_n$$

comprises a family of functions

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In other words, a natural transformation essentially tells us how to compose a chain of heteromorphisms.

When $n = 0$ and q is trivial, this is exactly the usual notion of natural transformation $\phi \colon f \Rightarrow g$ between functors.

The virtual double category of categories

The motivating example of a virtual double category is $\mathbf{Cat}$, the virtual double category whose objects are locally small categories, whose tight-cells are functors, whose loose-cells are distributors, and whose 2-cells are natural transformations.

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Other examples of virtual double categories include the virtual double categories $\mathbb{V}\text{-}\mathbf{Cat}$, of categories enriched in a monoidal category $\mathbb{V}$; $\mathbf{Cat}(\mathbb{E})$, of categories internal to a finitely complete category $\mathbb{E}$; as well as virtual double categories of fibred categories, indexed categories, monoidal categories, and so on.

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In fact, these virtual double categories are particularly well-behaved, having identity loose-cells, and restrictions of loose-cells along tight-cells. Such virtual double categories are known as virtual equipments.

Monads and loose-monads

Monads in a virtual double category

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A (tight) monad comprises a tight-cell $t\colon A \to A$, and 2-cells $\mu\colon tt \to t$ and $\eta\colon 1_A \Rightarrow t$ satisfying associativity and unitality axioms.

$$
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A & = & A & = & A \\
tt\downarrow & \mu & t\downarrow & \eta & \| \\
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A & = & A & = & A \\
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A & = & A & = & A
\end{array}
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A loose-monad comprises a loose-cell $t: A \nrightarrow A$, and 2-cells $\mu: t, t \to t$ and $\eta: \Rightarrow t$ satisfying associativity and unitality axioms.

$$
\begin{array}{ccc}
A \xleftarrow{t} A \xleftarrow{t} A & \qquad & A = A \\
\| \quad \mu \quad \| & & \| \quad \eta \quad \| \\
A \xleftarrow{\quad t \quad} A & & A \xleftarrow{t} A
\end{array}
$$

Monads and loose-monads in $\mathbf{Cat}$

A monad in $\mathbf{Cat}$ is simply an ordinary monad, i.e. a functor $t\colon A \to A$ equipped with natural transformations $\mu\colon tt \Rightarrow t$ and $\eta\colon 1_A \Rightarrow t$ satisfying associativity and unitality axioms.

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A loose-monad (a.k.a. promonad) in $\mathbf{Cat}$ comprises

1. a distributor $p\colon A \nrightarrow A$;
2. for each $f\colon x \to y$ in A, an element $\eta_f \in p(x,y)$;
3. for $f \in p(x,y)$ and $g \in p(y,z)$, an element $(f \mathbin{;} g) \in p(x,z)$;

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Indeed, every category A induces a canonical loose-monad $A(1, 1)$, whose underlying distributor is given by the hom-sets of A, whose unit is trivial, and whose multiplication is given by composition in A.

Loose-monads and identity-on-objects functors

In fact, every loose-monad p induces a category $\ll p \gg$, the collapse of p, defined by

$$|\ll p \gg| := |A| \qquad\qquad \ll p \gg (x, y) := p(x, y)$$

The collapse is equipped with an identity-on-objects functor $\amalg_p \colon A \to \ll p \gg$, which sends $f \colon x \to y$ in A to $\eta_f \colon x \to y$ in $\ll p \gg$.

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The collapse is equipped with an identity-on-objects functor $\underline{u}_p \colon A \to «p»$, which sends $f \colon x \to y$ in A to $\eta_f \colon x \to y$ in $«p»$.

Conversely, every identity-on-objects functor $f \colon A \to B$ induces a distributor $B(f,f) \colon A \nrightarrow A$, which is canonically equipped with the structure of a loose-monad. The unit is given by applying f, and the multiplication is given by functoriality of f.

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These assignments set up a bijection between loose-monads in $\mathbf{Cat}$, and identity-on-objects functors [Jus68].

Effective collapses in a virtual equipment

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Definition 7 ([Sch15])

Let (p, μ, η) be a loose-monad on an object A. An effective collapse of p comprises a diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ p\ } & A \\
{\scriptstyle \unlhd_A}\downarrow & {\scriptstyle \unlhd_p} & \downarrow{\scriptstyle \unlhd_A} \\
\ll p\gg & =\!=\!= & \ll p\gg
\end{array}
$$

satisfying $\ll p\gg(\unlhd_A, \unlhd_A) \cong p$, such that every loose-monad morphism from p into a loose-identity factors uniquely through $\unlhd_p$.

Full image factorisation

We observed earlier that every identity-on-objects functor $f \colon A \to B$ induces a loose-monad $B(f, f)$. In fact, $B(f, f)$ is a loose-monad even when f is not identity-on-objects.

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The effective collapse $\ll B(f, f) \gg$ of $B(f, f)$ is the full image of f, i.e. the category whose objects are those of A and whose morphisms $x \to y$ are morphisms $fx \to fy$ in B.

The full image forms a factorisation of f:

$$A \to \ll B(f, f) \gg \to B$$

where $A \to \ll B(f, f) \gg$ is identity-on-objects, and $\ll B(f, f) \gg$ is fully faithful.

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where $A \to \ll B(f, f) \gg$ is identity-on-objects, and $\ll B(f, f) \gg$ is fully faithful.

The (identity-on-objects, fully faithful) orthogonal factorisation system on $\mathbf{Cat}$ thus arises naturally from the theory of distributors [Woo85].

Relative monads

Monoids in multicategories

A multicategory [Lam69] is a generalisation of a category in which we permit morphisms with multiary domain (analogous to the 2-cells in a virtual double category).

We can define monoids internal to any multicategory, generalising the notion of monoid internal to a monoidal category.

Definition 8

Let $\mathbf{M}$ be a multicategory. A *monoid* in $\mathbf{M}$ comprises

1. an object M;
2. a multimorphism $\mu \colon M, M \to M$;
3. a multimorphism $\eta \colon \, \to M$,

satisfying associativity and unitality axioms.

Definition 9

Let $\mathbf{M}$ be a multicategory and let (M, μ_M, η_M) be a monoid in $\mathbf{M}$. An (M, μ_M, η_M)-*section* comprises a section–retraction pair $s \colon R \rightleftarrows M \colon r$ rendering the following diagram commutative.

$$
\begin{array}{ccc}
& R, R & \\
{}^{s,s}\swarrow & & \searrow^{s,s} \\
M, M & & M, M \\
\mu_M \downarrow & & \downarrow \mu_M \\
M \xrightarrow{\ r\ } R \xrightarrow{\ s\ } M & & \\
\underset{\eta_M}{\nwarrow} & & \nearrow_{\eta_M}
\end{array}
$$

Monoid sections II

Conceptually, a monoid section is a retract R of the carrier of a monoid M, for which the section morphism $s\colon R \to M$ satisfies the laws to be a monoid morphism, with respect to "tentative monoid structure" on R.

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It turns out that this suffices for R to itself be a monoid, whose multiplication and unit are inherited from M.

Proposition 10

Let $\mathbf{M}$ be a multicategory and let (M, μ_M, η_M) be a monoid in $\mathbf{M}$. An (M, μ_M, η_M)-section (R, s, r) endows R with a unique monoid structure such that s is a monoid morphism.

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Why is this interesting? It turns out that we can characterise relative monads in this way. This observation appears to be new even for non-relative monads.

Relative monads as monoid sections

Definition 11

A relative monad comprises a functor $t\colon A \to E$ along with a t-corepresentable $E(t, t)$-section.

Unwrapping this definition, we obtain the classical definition of a relative monad [ACU10], i.e. that a relative monad comprises

1. a functor $j\colon A \to E$, the *root*;
2. a functor $t\colon A \to E$, the *carrier*;
3. a natural transformation $\eta\colon j \Rightarrow t$, the *unit*;
4. a natural transformation $\dagger\colon E(j, t) \Rightarrow E(t, t)$, the *extension operator*,

satisfying unitality and associativity axioms.

When $j = 1$, this is equivalent to the usual definition of a monad.

Examples of relative monads

Relative monads are abundant in category theory.

- Monads.

Examples of relative monads

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- Monads.
- Partial monads.

Examples of relative monads

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- Monads.
- Partial monads.
- Graded monads [MU22].

Examples of relative monads

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- Monads.

- Partial monads.

- Graded monads [MU22].

- Cocontinuous monads on cocompletions (e.g. finitary monads on locally finitely presentable categories).

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- Monads arising from monad–theory correspondences [Ark22].

The loose-monad associated to a relative monad

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One reason is that we immediately obtain the following observation.

> ### Corollary 12
>
> Let T be a j-relative monad. The distributor $E(j,t)\colon A \nrightarrow A$ is equipped with the structure of a loose-monad $E(j,T)$, and $\dagger\colon E(j,t) \Rightarrow E(t,t)$ is a loose-monad morphism.

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Why is this nice? As we will see in the remainder of the talk, a relative monad T and its associated loose-monad $E(j,T)$ are strongly connected. The presentation of relative monads in terms of monoid sections emphasises this connection: in some sense, we can view $E(j,T)$ as encapsulating the fundamental structure of T.

Kleisli categories

Kleisli categories for relative monads

Just as for non-relative monads, there are two important categories associated to every relative monad.

Definition 13 ([ACU10])

Let $j\colon A \to E$ be a functor and let T be a j-relative monad. The Kleisli category of T is the category $\mathbf{Kl}(T)$ defined by

$$|\mathbf{Kl}(T)| := |A|$$
$$\mathbf{Kl}(T)(x, y) := E(jx, ty)$$

with identities and composition given as in the Kleisli category for a monad.

This is equipped with an inclusion functor $k_T\colon A \to \mathbf{Kl}(T)$.

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This definition may look reminiscent of an earlier one...

Kleisli categories via effective collapse

Theorem 14

Let T be a j-relative monad. The Kleisli category of T is precisely the effective collapse of the loose-monad $E(j, T)$.

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Why is this interesting?

- It allows us to capture what seems like an entirely concrete definition using distributors.
- The universal property of an effective collapse is *stronger* than that typically associated with a Kleisli category (namely an opalgebra object). This allows us to prove stronger theorems than we would otherwise be able to prove.

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- The universal property of an effective collapse is *stronger* than that typically associated with a Kleisli category (namely an opalgebra object). This allows us to prove stronger theorems than we would otherwise be able to prove.

- It justifies our perspective that $E(j, T)$ represents T in a suitable sense, since we can recover T from $\mathbf{Kl}(T)$ via its associated relative adjunction.

The pullback theorem

Categories of algebras

Definition 15 ([ACU10])

Let $j \colon A \to E$ be a functor and let T be a j-relative monad. A T-algebra is an object $e \in E$ equipped with a natural transformation $\rtimes \colon E(j, e) \Rightarrow E(t, e)$ that is compatible with the unit and extension operator of T.

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The category of algebras of T is the category $\mathbf{Alg}(T)$ whose objects are T-algebras and whose morphisms are morphisms in E preserving the algebra structure.

This is equipped with a forgetful functor $u_T: \mathbf{Alg}(T) \to E$.

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This is equipped with a forgetful functor $u_T\colon \mathbf{Alg}(T) \to E$.

When $j = 1$, this is equivalent to the usual definition of the category of algebras for a monad.

Relative adjunctions

The concept of relative adjunction is a generalisation of the concept of adjunction, where the domain of the left adjoint is permitted to be different to the codomain of the right adjoint.

Definition 16 ([Ulm68])

A *relative adjunction* comprises

1. a functor $j \colon A \to E$, the *root*;
2. a functor $\ell \colon A \to C$, the *left relative adjoint*;
3. a functor $r \colon C \to E$, the *right relative adjoint*;
4. an isomorphism of the form $C(\ell, 1) \cong E(j, r)$.
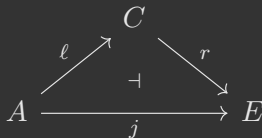
$$
\begin{array}{ccc}
 & C & \\
\ell \nearrow & \dashv & \searrow r \\
A & \xrightarrow{\quad j \quad} & E
\end{array}
$$

Examples of relative adjunctions

Relative adjunctions are abundant in category theory.

- Adjunctions.

Examples of relative adjunctions

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Examples of relative adjunctions

Relative adjunctions are abundant in category theory.

- Adjunctions.
- Partial adjunctions.
- Multi-adjunctions.
- Weighted colimits.
- Nerves.
- Algebraic theories and their various generalisations [Die74; Ark22].

Kleisli and Eilenberg–Moore relative adjunctions

Just as for non-relative monads, the Kleisli category and category of algebras associated to a relative monad T form relative adjunctions, which induce the relative monad T by composing the left relative adjoint with the right relative adjoint.

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Furthermore, these relative adjunctions satisfy universal properties amongst resolutions of T – i.e. relative adjunctions inducing T – which induce a canonical comparison functor $i_T \colon \mathbf{Kl}(T) \to \mathbf{Alg}(T)$.

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$$
\begin{array}{ccc}
& \mathbf{Kl}(T) & \\
{}^{k_T}\nearrow & & \searrow {}^{v_T} \\
& \dashv & \\
A & \xrightarrow{\quad j \quad} & E
\end{array}
\qquad\qquad
\begin{array}{ccc}
& \mathbf{Alg}(T) & \\
{}^{f_T}\nearrow & & \searrow {}^{u_T} \\
& \dashv & \\
A & \xrightarrow{\quad j \quad} & E
\end{array}
$$

Furthermore, these relative adjunctions satisfy universal properties amongst resolutions of T – i.e. relative adjunctions inducing T – which induce a canonical comparison functor $i_T \colon \mathbf{Kl}(T) \to \mathbf{Alg}(T)$.

As we shall see, this comparison functor exhibits a stronger universal property than is implied simply by the universal properties of $\mathbf{Kl}(T)$ or $\mathbf{Alg}(T)$ individually.

Exact pullbacks

Definition 17

A *exact pullback* of a span of functors $E \xleftarrow{j} A \xrightarrow{k} K$ comprises a span of a distributor and functor, as on the left, such that the diagram on the right commutes up to isomorphism,

$$
\begin{array}{ccc}
\bullet & \dashrightarrow^{i} & K \\
u \big\downarrow & & \big\uparrow k \\
E & \xleftarrow[j]{} & A
\end{array}
\qquad\qquad
\begin{array}{ccc}
\bullet & \xrightarrow{i} & K \\
E(1,u)\big\downarrow & \cong & \big\downarrow K(k,1) \\
E & \xrightarrow[E(j,1)]{} & A
\end{array}
$$

i.e. such that $i(k,1) \cong E(j,u)$, that is universal in the evident sense.
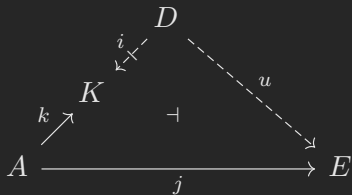
The exact pullback theorem

Theorem 18

Let $j: A \to E$ be a dense functor and let T be a j-relative monad. The following diagram is an exact pullback.

$$\begin{array}{ccc} \mathbf{Kl}(T) & \xrightarrow{\;i_T\;} & \mathbf{Alg}(T) \\ \scriptstyle{k_T}\big\uparrow & & \big\downarrow\scriptstyle{u_T} \\ A & \xrightarrow[\;\;j\;\;]{} & E \end{array}$$
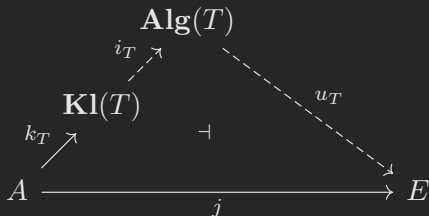
This is striking, because it identifies a nontrivial universal property that connects $\mathbf{Kl}(T)$ and $\mathbf{Alg}(T)$.

Universal relative right adjoints

Intuitively, we can view an exact pullback as a universal right j-adjoint.

The exact pullback theorem therefore states that the following forms a universal right j-adjoint.

Presheaf categories

Definition 19

Let A be a small category. The category of presheaves on A is the functor category $\widehat{A} := [A^{\mathrm{op}}, \mathbf{Set}]$. Denote by $\text{よ}_A \colon A \to \widehat{A}$ the Yoneda embedding, defined by

$$\text{よ}_A(a) := A(-, a)$$

Presheaf categories

Definition 19

Let A be a small category. The category of presheaves on A is the functor category $\widehat{A} := [A^{\mathrm{op}}, \mathbf{Set}]$. Denote by $\sideset{}{_A}\And : A \to \widehat{A}$ the Yoneda embedding, defined by

$$\sideset{}{_A}\And(a) := A(-, a)$$

We can reformulate the Yoneda lemma in terms of a universal property involving distributors.

Lemma 20

The Yoneda embedding $\widehat{A}(\sideset{}{_A}\And, 1)$ induces a bijection between functors $B \to \widehat{A}$ and distributors $B \nrightarrow A$.

Nerves

Definition 21

For any functor $f \colon A \to B$ from a small category, there is a functor $n_f \colon B \to \widehat{A}$, the nerve of f, defined by

$$n_f(b) := B(f-, b)$$

The nerve is the functor corresponding, via the bijection on the previous slide, to the corepresentable distributor $B(f, 1) \colon B \nrightarrow A$.

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The nerve of f is right relative adjoint to f.

The pullback theorem I

In the presence of categories of presheaves, we may reformulate the universal property of an exact pullback into one involving only functors (rather than distributors). This allows us to easily give a concrete description.

Theorem 22

In **Cat***, the exact pullback of a span $E \xleftarrow{j} A \xrightarrow{k} K$, where A and K are small, is given by the following pullback.*

$$
\begin{array}{ccc}
\bullet & \xdashrightarrow{\;\;i\;\;} & \widehat{K} \\
u \Big\downarrow & \lrcorner & \Big\downarrow \widehat{k} \\
E & \xrightarrow{\;\;n_j\;\;} & \widehat{A}
\end{array}
$$

The pullback theorem II

Corollary 23

Let $j: A \to E$ be a dense functor and let T be a j-relative monad. The following diagram is a pullback in **Cat**.

$$
\begin{array}{ccc}
\mathbf{Alg}(T) & \lhook\joinrel\longrightarrow & \widehat{\mathbf{Kl}(T)} \\
{\scriptstyle u_T}\big\downarrow{\scriptstyle \lrcorner} & & \big\downarrow{\scriptstyle \widehat{k_T}} \\
E & \underset{n_j}{\lhook\joinrel\longrightarrow} & \widehat{A}
\end{array}
$$

Consequently, the comparison functor $i_T: \mathbf{Kl}(T) \hookrightarrow \mathbf{Alg}(T)$ is dense.

The algebras for a relative monad may thus be seen as a free cocompletion of the free algebras.

Non-relative case

When $j = 1$, this theorem is known (though less well known than it perhaps ought to be).

Theorem 24 (Linton)

Let T be a monad on a category A. The following diagram is a pullback in $\mathbf{Cat}$.

$$
\begin{array}{ccc}
\mathbf{Alg}(T) & \hookrightarrow & \widehat{\mathbf{Kl}(T)} \\
{\scriptstyle u_T}\downarrow \quad {\scriptstyle \lrcorner} & & \downarrow {\scriptstyle k_T{}^*} \\
A & \xrightarrow{\ \ \text{よ}_A\ \ } & \widehat{A}
\end{array}
$$

Consequences

Algebraic theories and relative monads

One of our motivations for studying relative monads is their connection to algebraic theories and their generalisations.

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Definition 25 (Lawvere)

Denote by $\mathbb{F}$ the free category with strict finite coproducts on a single object. A *finitary algebraic theory* is an identity-on-objects functor from $\mathbb{F}$ that preserves finite coproducts.

Algebraic theories and relative monads

One of our motivations for studying relative monads is their connection to algebraic theories and their generalisations.

Definition 25 (Lawvere)

Denote by $\mathbb{F}$ the free category with strict finite coproducts on a single object. A *finitary algebraic theory* is an identity-on-objects functor from $\mathbb{F}$ that preserves finite coproducts.

Theorem 26

There is an isomorphism between the category of finitary algebraic theories and the category of $(\mathbb{F} \to \mathbf{Set})$-relative monads.

More specifically, every algebraic theory is the Kleisli inclusion of a relative monad [Ark22].

Models and algebras

The pullback theorem establishes that the correspondence between algebraic theories and relative monads commutes with the process of taking models and algebras respectively.

Corollary 27

Let $\ell \colon \mathbb{F} \to L$ be a finitary algebraic theory. The category of algebras for the induced relative monad is given by the following pullback in $\mathbf{Cat}$.

$$
\begin{array}{ccc}
\mathbf{Cart}(L^{\mathrm{op}}, \mathbf{Set}) & \lhook\joinrel\longrightarrow & \widehat{L} \\
{\scriptstyle \mathbf{Cart}(\ell^{\mathrm{op}}, \mathbf{Set})} \big\downarrow & \lrcorner & \big\downarrow {\scriptstyle \widehat{\ell}} \\
\mathbf{Cart}(\mathbb{F}^{\mathrm{op}}, \mathbf{Set}) & \lhook\joinrel\longrightarrow & \widehat{\mathbb{F}}
\end{array}
$$

Cocontinuous monads and relative monads

Another motivation for studying relative monads is their connection to cocontinuous monads.

Theorem 28

Let Φ be a class of colimits. There is an equivalence between the category of Φ-cocontinuous monads on $\Phi(A)$ and the category of $(A \to \Phi(A))$-relative monads, and this commutes with the process of taking algebras.

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Let Φ be a class of colimits. There is an equivalence between the category of Φ-cocontinuous monads on $\Phi(A)$ and the category of $(A \to \Phi(A))$-relative monads, and this commutes with the process of taking algebras.

Corollary 29

Let A be a small, finitely cocomplete category. There is an equivalence between the category of finitary monads on $\mathbf{Ind}(A)$ and the category of $(A \to \mathbf{Ind}(A))$-relative monads, and this commutes with the process of taking algebras.

Locally presentable categories of algebras

Corollary 30

Let T be a finitary monad on a locally finitely presentable category. Then its category of algebras is also locally finitely presentable.

Locally presentable categories of algebras

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Let T be a finitary monad on a locally finitely presentable category. Then its category of algebras is also locally finitely presentable.

Proof sketch. We have the following pullback in $\mathbf{Cat}$.

$$\begin{array}{ccc}
\mathbf{Alg}(T) & \hookrightarrow & \widehat{\mathbf{Kl}(T)} \\
u_T \downarrow \quad {\scriptstyle \lrcorner} & & \downarrow \widehat{k_T} \\
\mathbf{Ind}(A) & \hookrightarrow & \widehat{A}
\end{array}$$

The functors $\mathbf{Ind}(A) \to \widehat{A}$ and $\widehat{\mathbf{Kl}(T)} \to \widehat{A}$ are both finitary right adjoints between locally finitely presentable categories. Thus, so are the two projection functors [Bir84].

Summary

- There is a strong connection between the theory of relative monads and the theory of distributors.
- The pullback theorem for monads generalises to a pullback theorem for relative monads with dense roots.
- This has fruitful connections to the theory of algebraic theories and cocontinuous monads.

We're currently in the process of writing a paper on this topic. In the meantime, if you found this talk interesting, you may also be interested in:

1. *Monadic and Higher-Order Structure* [Ark22]
2. *The formal theory of relative monads* [AM23a]
3. *Relative monadicity* [AM23b]

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